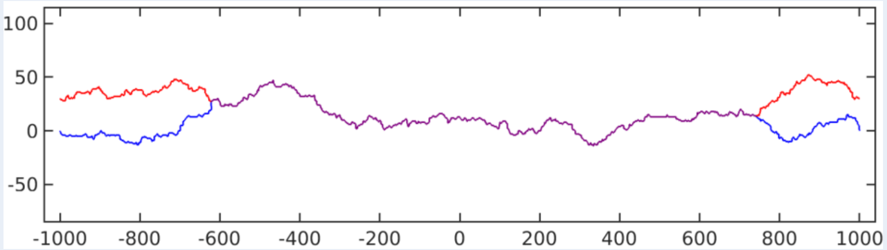
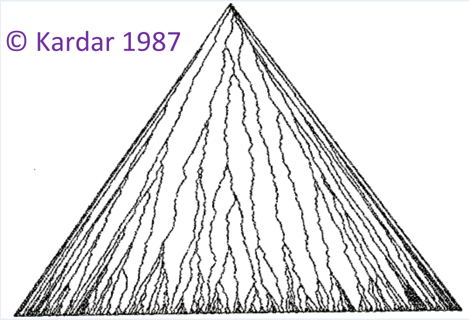
Coalescence of geodesics and the BKS midpoint problem in planar first-passage percolation





The authors find a fast route in the random environment

Ron Peled, Tel Aviv University,
on sabbatical at the IAS and Princeton University

Joint work with Barbara Dembin and Dor Elboim

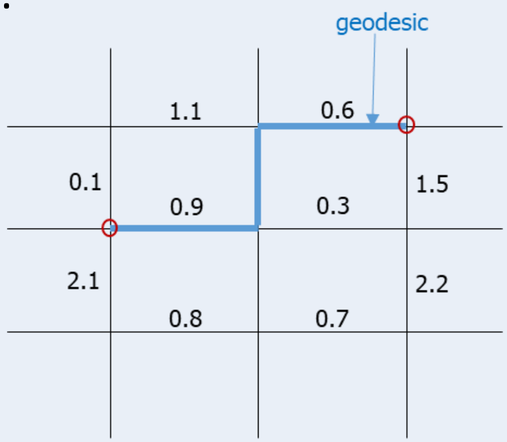
123rd Statistical Mechanics Conference,
in honor of Matthew Fisher and Charles Newman
December 18, 2022

Planar first-passage percolation

- Idea: Random perturbation of Euclidean geometry, formed by a random media with short-range correlations (Hammersley-Welsh 65).
  In this talk we focus on the discrete planar setting, working on the lattice $\mathbb{Z}^2$.
- Edge weights: Independent and identically distributed non-negative $(\tau_e)_{e\in E(\mathbb{Z}^2)}$.
  In this talk assume (partly for simplicity) that their common distribution is absolutely continuous and has compact support in $(0,\infty)$.
  E.g., $\tau_e \sim \mathrm{Uniform}[1,2]$.
- Passage time: A random metric $T_{u,v}$ on $\mathbb{Z}^2$ given by

$$T_{u,v} := \min \sum_{e\in p} \tau_e$$

  with the minimum over paths p connecting u and v.

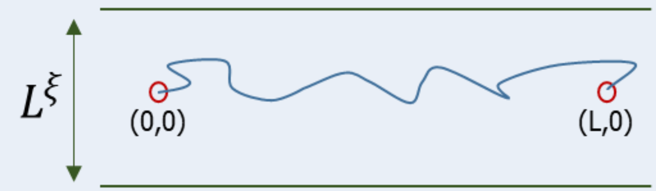


- Geodesic: A path p realizing $T_{u,v}$, denoted $\gamma_{u,v}$.
  Existence and uniqueness guaranteed by absolute continuity assumption.
- Goal: Understand the large-scale properties of the metric T.
  In particular, understand long geodesics.

Basic predictions

- For a point $v \in \mathbb{R}^2$ and $L > 0$, consider the passage time $T_{\mathbf{0},Lv}$ and geodesic $\gamma_{\mathbf{0},Lv}$ (abbreviating $(0,0)$ to $\mathbf{0}$ and rounding Lv to the closest lattice point of ℤ^2).
- Basic predictions: as $L \to \infty$,

$$\mathbb{E}(T_{\mathbf{0},Lv}) = \mu(v)L - c_1 L^\chi(1 + o(1))$$
$$Std(T_{\mathbf{0},Lv}) = c_2 L^\chi(1 + o(1))$$



  the transversal fluctuations of $\gamma_{\mathbf{0},Lv}$ are of order L^ξ.

  The model is in the KPZ universality class with $\chi = \frac{1}{3}$ and $\xi = \frac{2}{3}$

  (Huse-Henley 85, Kardar 85, Huse-Henley-D.S.Fisher 85, Kardar-Parisi-Zhang 86)
- Limit norm: $\mu(v)$ is a (deterministic) norm on $\mathbb{R}^2$, almost surely given by

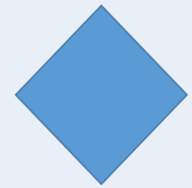
$$\mu(v) = \lim_{L\to\infty} \frac{T_{\mathbf{0},Lv}}{L}$$

- Limit shape: unit ball $B \coloneqq \{v \in \mathbb{R}^2 : \mu(v) \leq 1\}$ strictly convex.
  Specific shape of B depends on the edge weight distribution.
  Unclear whether it is ever a Euclidean ball.

$B =$ 

Rigorous results

- Norm: $\mu(v)$ is well defined. Not proved that its unit ball B is strictly convex! Not even proved that B is never the ℓ_1 or ℓ_∞ ball!

$B =$  ?

$B =$  ?

- Standard deviation:

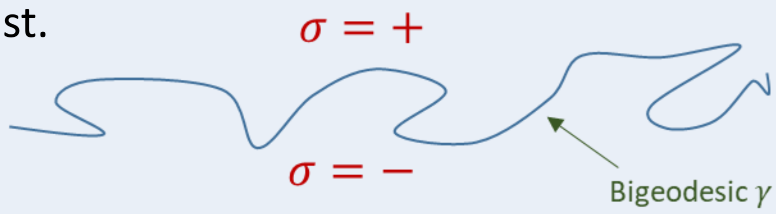
$$Std(T_{\mathbf{0},Lv}) \geq c\sqrt{\log L} \quad \text{(Newman-Piza 95)}$$

$$Std(T_{\mathbf{0},Lv}) \leq c\sqrt{\frac{L}{\log L}} \quad \text{(Benjamini-Kalai-Schramm 02)}$$

- Transversal fluctuations: version of $\xi \geq \frac{1}{3}$ (Licea-Newman-Piza 96)
  No proof that the transversal fluctuations are of order $o(L)$!
- Book of Auffinger-Damron-Hanson 15 surveys the rigorous state-of-the-art. Many basic questions remain open.
- Detailed understanding available for a related integrable model: Directed last-passage percolation (with specific edge weight distributions). However, no integrable first-passage percolation model is known.

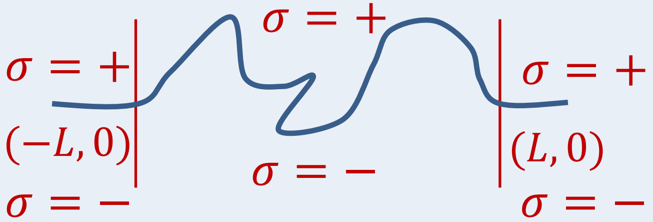
Disordered systems perspective

- Disordered ferromagnet: $\tau = (\tau_e)_{e \in E(\mathbb{Z}^d)}$ IID non-negative edge weights as before.
  The disordered Ising ferromagnet is the model on $\sigma: \mathbb{Z}^d \to \{-1,1\}$ with formal Hamiltonian
  $$H^\tau(\sigma) := -\sum_{e=\{u,v\} \in E(\mathbb{Z}^d)} \tau_e \sigma_u \sigma_v$$
- Ground configurations: Configurations $\sigma: \mathbb{Z}^d \to \{-1,1\}$ whose energy cannot be lowered by flipping finitely many spins.
  The constant configurations $\sigma \equiv +$ and $\sigma \equiv -$ are ground configurations.
- Basic challenge: Are there non-constant ground configurations?
- When $d = 2$, their existence is equivalent to the existence of bigeodesics in the first-passage percolation model with weights τ (Licea-Newman 96).
  Bigeodesic: a doubly-infinite path for which every finite segment is a geodesic.
  When $d = 2$, it is conjectured that bigeodesics do not exist and hence non-constant ground configurations do not exist.



Dobrushin boundary conditions and the Benjamini-Kalai-Schramm midpoint problem

- Dobrushin boundary conditions: A natural way to obtain a non-constant ground configuration is to consider the infinite-volume subsequential limit of ground configurations in finite domains with Dobrushin boundary conditions (+ spins above, - spins below).
  For $d = 2$, it is expected to yield a constant configuration, as the finite-volume interface fluctuates away.



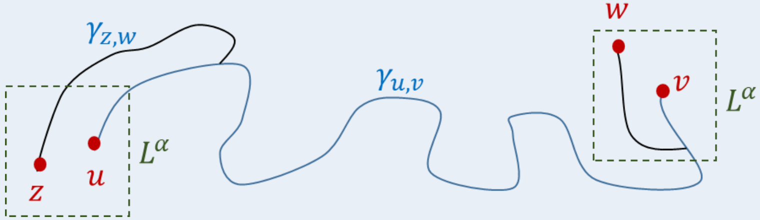
- BKS midpoint problem: Analysis of finite-volume interfaces with Dobrushin boundary conditions is thus related to the following midpoint problem: Prove that

$$\lim_{\substack{|u-v|\to\infty \\ u,v\in\mathbb{Z}^2}} \mathbb{P}\left(\gamma_{u,v} \text{ passes within distance 1 of } \frac{u+v}{2}\right) = 0$$

- For $d = 2$, this was proved in great generality by Ahlberg-Hoffman 16, following Damron-Hanson 15 who assumed the differentiability of the limit shape boundary. Both proofs are non-quantitative.
- The BKS midpoint problem can also be thought of as bounding the influence of specific edges on the passage time between u and v. This was the BKS perspective.

Results (coalescence of geodesics and BKS midpoint problem)

- Limit shape assumption: We assume that the limit shape has more than 32 extreme points. This assumption seems mild and we can verify that it holds for a class of edge weight distributions (perturbations of a deterministic edge weight).
- Theorem (Dembin-Elboim-P. 22, "Coalescence exponent $\geq 1/8$"):
  Let $u, v \in \mathbb{Z}^2$ and set $L = |u - v|$. Then, for every $0 < \alpha < 1/8$,
  $$\mathbb{P}\left(\exists z, w \text{ with } \max\{|z-u|, |w-v|\} \leq L^\alpha \text{ s.t. } |\gamma_{z,w} \Delta \gamma_{u,v}| > \frac{L}{\log L}\right) \leq CL^{-c(\alpha)}$$
- First quantitative proof for coalescence of geodesics, except Alexander 20 who used very strong assumptions, currently verified only in exactly-solvable models.



- Presumably, the coalescence exponent equals $\xi = \frac{2}{3}$ in two dimensions.
- Corollary (Dembin-Elboim-P. 22, quantitative BKS midpoint problem):
  Let $u, v \in \mathbb{Z}^2$ and set $L = |u - v|$. Then,
  $$\mathbb{P}\left(\gamma_{u,v} \text{ passes within distance 1 of } \frac{u+v}{2}\right) \leq CL^{-c}$$